

# A MODEL OF ELECTROSTATIC INTERACTION IN ONE-, TWO- AND THREE-DIMENSIONAL SPACES

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## Abstract

In this paper an attempt to build a model of a network of masses connected by springs is taken, and to create a change in this network, which would be an equivalent to the charges located in space. We study here the interdependence of energy and distance between charges, and we try to find such a solution to the model, which would be most appropriate to describe the electrostatic field in the real world.

## 1.1 Introduction

Electrostatic interaction of charges seems perfectly known. We know that the force acting on two charged objects is directly proportional to the product of charges and inversely proportional to the square of distance between them:

$$F \sim \frac{Qq}{r^2} \quad (1)$$

Because force is a distance derivative of potential energy,

$$F = \frac{dV}{dr} \quad (2)$$

we can say that the energy, exact to the integration constant, equals:

$$V \sim \int \frac{Qq}{r^2} dr \quad (3)$$

$$V \sim \frac{1}{r} \quad (4)$$

The interdependence between force, energy and interaction is well known and proved by experiment.

But here we arrive at a question about the Coulomb potential in 1D and 2D cases. We are unable to check it experimentally, because we live in a 3-dimensional space. So how can we verify our theoretical assumptions in these cases? This paper is an attempt to build a model of this interaction and to study its behaviour according to the number of dimensions. If we manage to build a model, whose behaviour in 3D space corresponds to (4), there will be a hypothesis that it is also true in 1D and 2D spaces.

## 1.2 Expected behaviour of the model

We know that in the 3D case force is inversely proportional to the square of distance. We also know that the total stream of field is constant. Thus we get:

$$\sigma \sim F_1 4\pi r_1^2 \quad (5)$$

After multiplying the distance by  $a$ , that stream is defined as:

$$\sigma \sim F_2 4\pi r_2^2 \quad (6)$$

$$r_2 = ar_1 \quad (7)$$

$$\frac{F_2}{F_1} = \frac{1}{a^2} \frac{4\pi r_1^2}{4\pi r_1^2} \quad (8)$$

which confirms the interdependence:

$$F_{3D} \sim \frac{1}{r^2} \quad (9)$$

In the 2D case force “distributes itself” on the circumference of the circle, which defines the stream as:

$$\sigma \sim F_1 2\pi r_1 \quad (10)$$

After multiplying the distance by  $a$ :

$$\sigma \sim F_2 2\pi r_2 \quad (11)$$

$$r_2 = ar_1 \quad (12)$$

$$\frac{F_2}{F_1} = \frac{1}{a} \frac{2\pi r_1}{2\pi r_1} \quad (13)$$

therefore in the 2D case

$$F_{2D} \sim \frac{1}{r} \quad (14)$$

In the 1D case the interaction spreads in a straight line, which means that force does not depend on distance.

$$F_{1D} \sim r^0 \quad (15)$$

Distance integrating the forces we got, we receive the expected interdependences of potential energy and distance, in the 1D, 2D and 3D cases.

$$F_{1D} \sim r^0 \quad (16)$$

$$V_{1D} \sim \int r^0 dr \quad (17)$$

$$V_{1D} \sim r \quad (18)$$

$$F_{2D} \sim \frac{1}{r} \quad (19)$$

$$V_{2D} \sim \int \frac{dr}{r} \quad (20)$$

$$V_{2D} \sim \ln r \quad (21)$$

$$F_{3D} \sim \frac{1}{r^2} \quad (22)$$

$$V_{3D} \sim \int \frac{dr}{r^2} \quad (23)$$

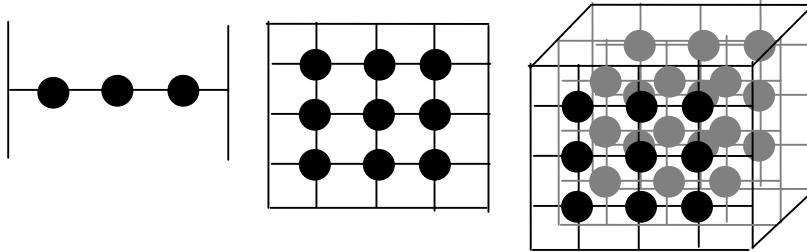
$$V_{3D} \sim \frac{1}{r} \quad (24)$$

Summarizing, the model of interactions that will be accurate and appropriate in this interaction should have the following interdependences:

	$F(r)$	$V(r)$
1D	$\sim r^0$	$\sim r$
2D	$\sim \frac{1}{r}$	$\sim \ln r$
3D	$\sim \frac{1}{r^2}$	$\sim \frac{1}{r}$

## 2.1 Suggested model

A network with an appropriate number of dimensions is built: in the 3D case it will be a cubic network, in 2D case a rectangular network, and in 1D case a sequence of masses in a line. Those masses are connected by springs along the co-ordinate axes and to the walls:



**Fig1.** A model of the “vacuum” for the 1D, 2D and 3D models.

Such arrangement for our future interactions shall be called a vacuum, and its basic energy – vacuum energy. Let’s state the distance between the walls as  $D$ , and the number of atoms on the axis as  $n$ . Additionally a few assumptions must be taken:

- the boundary conditions cannot influence the results of the measurements, which means we assume that:

$$D \rightarrow \infty \quad (25)$$

- the distances between masses are minimal, which increases the accuracy of the results and the suitability of the model to the field description. It means that the number of masses is maximal

$$n \rightarrow \infty \quad (26)$$

and in the meantime the distance between them:

$$\frac{D}{n+1} \rightarrow 0 \quad (27)$$

- all actions in such a model should be independent on the choice of directions (no direction should be preferred), which is, unfortunately, impossible to accomplish in such a model in reality. The idealised model assumes that each mass is connected with every other, which means that all directions are realized. But it is a situation impossible to realize in numerical simulation.
- we give the springs a specific property assuming that their static length equals 0. It defines the force of interactions of masses connected by springs:

$$F_{1 \leftrightarrow 2} = k(x_2 - x_1) \quad (28)$$

and:

$$V_{1 \leftrightarrow 2} = \frac{k}{2}(x_2 - x_1)^2 \quad (29)$$

Assuming on this basis that we have a network with  $D_x$ ,  $D_y$  and  $D_z$  dimensions, with  $n_x$ ,  $n_y$  and  $n_z$  atoms respectively in each dimension. The static energy of the system:

$$E_0 = \frac{k}{2} \sum_{\alpha=x,y,z} (n_\alpha + 1) \left( \frac{D_\alpha}{n_\alpha + 1} \right)^2 \quad (30)$$

Finally we get the energy of the initial state (here: vacuum energy):

$$E_0 = \frac{k}{2} \sum_{\alpha=x,y,z} \frac{D_\alpha^2}{n_\alpha + 1} \quad (31)$$

## 2.2 Definition of the charge.

In such “field” we have to place charges, which could interact. Therefore we choose two groups of masses. In one case we diminish the distances between these masses – let’s call it a positive charge. In the other we increase the distances, stretching the springs – and it will be our negative charge. A few questions arise that we will be trying to solve:

- What do we mean by unlike charges with the same arithmetical value and opposite senses and is there some symmetry between them?
- Is the value of charges dependent on the quantity of so transformed masses, or on the changes of springs’ length?
- What shape should this deformation have?
- Do we permit extreme network deformations (change in the order of masses) and how to prevent them?

We will try to answer these questions in following studies, but first let’s give a careful consideration to the 1D model.

## 3.1 Analytical solution of the 1D model

It is possible to solve the 1D case basing only on theoretical studies, therefore I won’t quote here the results of computer simulations, in the meantime stating that they are wholly consistent with the results of theoretical studies.

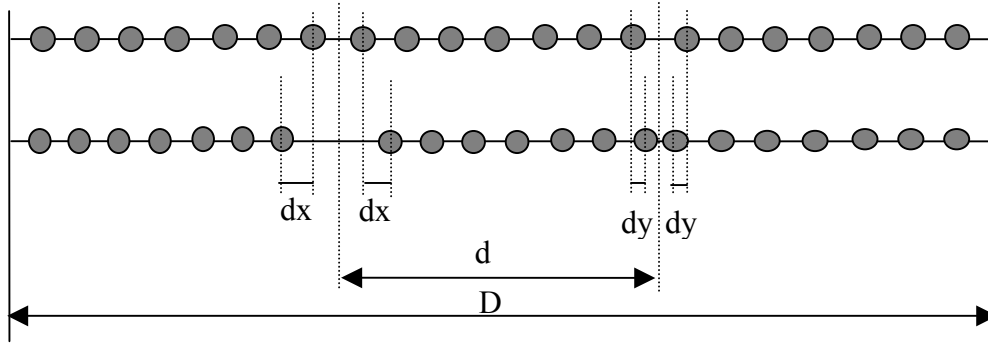
It must be said, that irrespective of the number of dimensions we can assume two general strategies of creating charges:

- a. A charge created by changing the distance between masses is permanently connected to its place of origin, and cannot shift inside the network.
- b. The deformation has the possibility to occupy the optimal place in space, the only limitation being the distance between masses.

Both models produce different interdependences between energy and distance so we have to consider them separately.

### 3.1.a Anchored model

We choose two springs and we denote the distance between their centres as  $d$ . We stretch one by  $dx$ , and the other by  $dy$  (where a positive value of  $dx$  means contracting, and a negative value - stretching). The distance between walls is  $D$ . The length of springs before the creation of "charges" is  $s$ . The elastic coefficient is  $k$ .



**Fig2.** A model of charges in 1D

Let AF be the section on which we consider the situation, BC – first charge, DE – second charge. We calculate following values:

$$n_1 = \frac{D-d-s}{2s} - \text{number of springs on AB} \quad (32)$$

$$n_2 = \frac{d-s}{s} - \text{number of springs on CD} \quad (33)$$

$$n_3 = \frac{D-d-s}{2s} - \text{number of springs on EF} \quad (34)$$

$$AB = \frac{D-d-s}{2} + \frac{dx}{2} \quad (35)$$

$$CD = d-s + \frac{dx}{2} + \frac{dy}{2} \quad (36)$$

$$EF = \frac{D-d-s}{2} + \frac{dy}{2} \quad (37)$$

Energy equals then:

$$E = \frac{k}{2} \left( n_1 \left( \frac{AB}{n_1} \right)^2 + n_2 \left( \frac{CD}{n_2} \right)^2 + n_3 \left( \frac{EF}{n_3} \right)^2 \right) \quad (38)$$

After inserting the interdependence (1-6) and subtracting the vacuum energy we get:

$$E - E_0 = \frac{k}{2} \left( dx^2 + dy^2 + s \frac{dx^2 + dy^2}{2(D-d-s)} + s \frac{(dx+dy)^2}{4(d-s)} \right) \quad (39)$$

If we assume additionally that the positive charge has the same value as the negative charge, we stretch one spring equally as we contract the other one:

$$dx = -dy \quad (40)$$

and we ignore the boundary conditions

$$D \rightarrow \infty \quad (41)$$

then the last element of the equation disappears and we get:

$$E - E_0 = \frac{k}{2} \left( dx^2 + dy^2 + s \frac{dx^2 + dy^2}{2(D-d-s)} \right) \quad (42)$$

where:

$$s \frac{dx^2 + dy^2}{2(D-d-s)} \rightarrow 0 \quad (43)$$

$$E - E_0 = \frac{k}{2} dx^2 + dy^2 \quad (44)$$

which makes energy independent on distance:

$$E_{seen} \sim r^0 \quad (45)$$

However, if we ignore the condition of equality of charges, and leave the space infinite:

$$dx \neq -dy \quad (46)$$

$$D \rightarrow \infty \quad (47)$$

Thus we get:

$$s \frac{dx^2 + dy^2}{2(D-d-s)} \rightarrow 0 \quad (48)$$

$$E - E_0 = \frac{k}{2} \left( dx^2 + dy^2 + s \frac{(dx+dy)^2}{4(d-s)} \right) \quad (49)$$

what produces the interdependence:

$$E_{seen} \sim \frac{1}{r} \quad (50)$$

### 3.1.b Unanchored model

Let's study an unanchored model. The only assumption we make is the distance between the atoms inside the charge. All designations are like in the previous case. To find the state of equilibrium, we search for the a, b, c minimum of the sum:

$$E(a) + E(b) + E(c) + \frac{k}{2} \left( (s-dx)^2 + (s-dy)^2 \right) \quad (51)$$

where:

$$E(a) = \frac{k}{2} n_1 \left( \frac{a - \frac{s}{2} + \frac{dx}{2}}{n_1} \right)^2 \quad (52)$$

$$E(b) = \frac{k}{2} n_2 \left( \frac{a - s + \frac{dx}{2} + \frac{dy}{2}}{n_2} \right)^2 \quad (53)$$

$$E(c) = \frac{k}{2} n_3 \left( \frac{c - \frac{s}{2} + \frac{dy}{2}}{n_3} \right)^2 \quad (54)$$

Therefore we search for the minimal value of the expression:

$$\frac{x^2}{n_1} + \frac{y^2}{n_2} + \frac{z^2}{n_3} \quad (55)$$

true to the condition:

$$x + y + z = \alpha \quad (56)$$

Therefore we study the function:

$$f(x, y, z) = \frac{x^2}{n_1} + \frac{y^2}{n_2} + \frac{z^2}{n_3} + \lambda(x + y + z - \alpha) \quad (57)$$

and calculate the x, y, z derivatives, comparing them to 0. We get a set of equations:

$$\left\{ \begin{array}{l} \frac{2x}{n_1} = -\lambda \\ \frac{2y}{n_2} = -\lambda \\ \frac{2z}{n_3} = -\lambda \\ x + y + z = \alpha \end{array} \right. \quad (58)$$

Its solution is the sought minimum.

$$\left\{ \begin{array}{l} x = \frac{n_1 \alpha}{n_1 + n_2 + n_3} \\ y = \frac{n_2 \alpha}{n_1 + n_2 + n_3} \\ z = \frac{n_3 \alpha}{n_1 + n_2 + n_3} \end{array} \right. \quad (59)$$

Because:

$$\begin{cases} \alpha = D - 2s + dx + dy \\ x = a - \frac{s}{2} + \frac{dx}{2} \\ y = b - s + \frac{dx}{2} + \frac{dy}{2} \\ z = c - \frac{s}{2} + \frac{dy}{2} \end{cases} \quad (60)$$

the arrangement reaches the energetical minimum, when

$$E(a) = \frac{k}{2} n_1 \left( \frac{D - 2s + dx + dy}{n_1 + n_2 + n_3} \right)^2 \quad (61)$$

$$E(b) = \frac{k}{2} n_2 \left( \frac{D - 2s + dx + dy}{n_1 + n_2 + n_3} \right)^2 \quad (62)$$

$$E(c) = \frac{k}{2} n_3 \left( \frac{D - 2s + dx + dy}{n_1 + n_2 + n_3} \right)^2 \quad (63)$$

Finally we get:

$$E_{seen} = E - E_0 = \frac{k}{2} \left( \frac{(dx + dy)^2}{D - 2s} + dx^2 + dy^2 \right) \quad (64)$$

That interdependence implies that the energy is independent on  $d$  – it is constant.

#### 4.1 2D model – a sketch of the algorithm

Let's proceed now to the 2D model. In this case the analytical methods are not so efficient anymore, therefore we have to build a model basing on computer simulation.

First we spread the masses equally. The forces acting on the atom located on  $\vec{x}(i, j)$  from 4 neighbouring masses can be defined as:

$$\vec{F}(i, j) = k \left| \vec{x}(i \pm 1, j \pm 1) - \vec{x}(i, j) \right| \quad (65)$$

The movement of such a model can be described by the following equations

$$\dot{\vec{x}} = \frac{\vec{p}}{m} \quad (66)$$

$$\dot{\vec{p}} = -\frac{\partial V}{\partial \vec{x}} \quad (67)$$

Choosing an appropriately small  $\Delta t$  we can state:

$$x(i, j)_+ = \frac{p(i, j)}{m} \Delta t \quad (68)$$

$$p(i, j)_+ = F(i, j) \Delta t \quad (69)$$

For the whole arrangement to descend to the state of equilibrium we have to introduce dumping, by constant diminishing the momentum by a given percent.

To the whole arrangement we can introduce charges in two ways:



- firstly stating the distance between atoms – then the arrangement begins to oscillate strongly, and after some time descends to the state of equilibrium

- creating charges one by one, giving the network the possibility to adjust itself to the current situation – the oscillations don't really exist, the arrangement very quickly reaches the energetic minimum.

Thus we get the algorithm that simulates the behaviour of the network and allows energy measurements.

#### 4.2 Reaching the interdependence between energy and distance in 2D model

The first conclusion we reach during the simulation is the negation of the known assumption about the independency on the choice of directions. It turns out that the choice of the location of the charges in the network influences the final behaviour of the model. Chart 1. shows energy as a function of distance in 3 cases:

*a* – the charges are located on a line parallel to the directions of the network

*b* - the charges are located at an 45° angle to the directions of the network

*c* – the charges are located as in *a* , but with no symmetry to the edges of the network

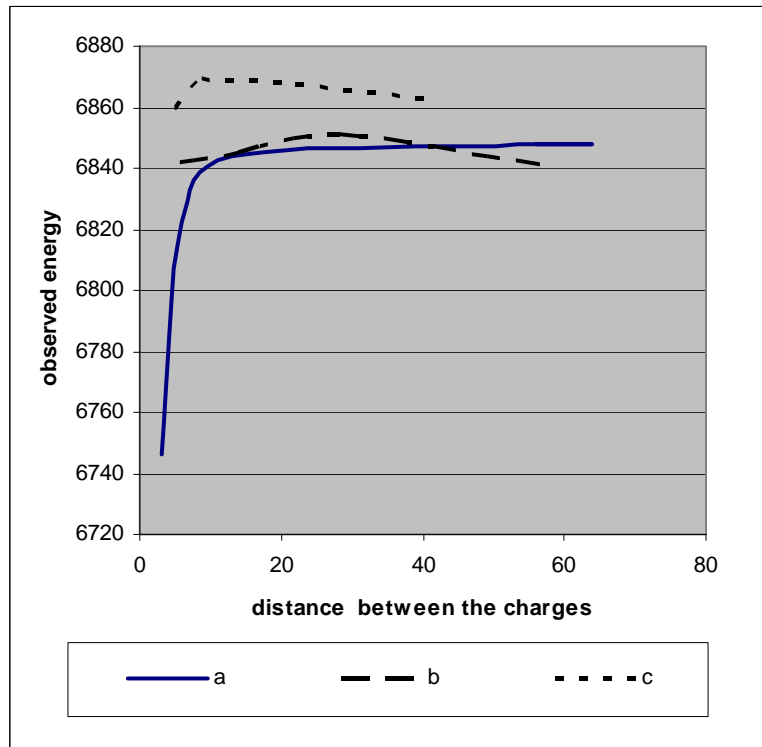


Chart 1. Measurements of the 2D model

The interdependence *c* shows that the edges of the network also have a noticeable influence on the results.

It must be noted, that expanding the network doesn't have sense because of computation capacities. That poses a question whether the numerical method is efficient enough, and is applicable to the 3D model.

It is quite difficult to summarize the results of the study, but we may assume that the mean curve can be approximated as a parabole, what implies the interdependence:

$$E_{2D} \sim \sqrt{x} \tag{70}$$

Of course it is a so-called anchored model.

#### 5.1 Hitherto results

Let's compare now the interdependences between energy and distance with those assumed at the beginning and expected as a check of the model's appropriateness to describe an electrostatic field.

	Expected	Received in the model	
		Anchored	unanchored
1D	$\sim r$	$\sim r^0 \vee \sim \frac{1}{r}$	$\sim r^0$
2D	$\sim \ln r$	$\sim \sqrt{r}$	??
3D	$\sim \frac{1}{r}$	??	??

A serious obstacle in studying the 3D model is a need to find such an algorithm, which will render possible a fast calculation of the network movements with a big amount of atoms. A calculation of a 3D model with 200x200x200 atoms will take 200x200 200 times more time, comparing to the 2D model (about 40-50s x 200 = 130-170 min on a 1GHz processor). It must also be said, that in such dimensions – of several hundred atoms – the influence of boundary conditions is too big to be ignored. Expanding the network 4 times means 64 times more calculation time, what is an overwhelming barrier to an average PC.

Calculating the distance derivatives of energy:

	$V(r)$	$F(r)$
1D	$\sim r^0 \vee \sim \frac{1}{r}$	$\sim \frac{1}{r} \sim \frac{1}{r^2}$
2D	$\sim \sqrt{r}$	$\sim \frac{1}{\sqrt{r}}$

$$\frac{1}{r} |_{1D} \xrightarrow{\cdot\sqrt{r}} \frac{1}{\sqrt{r}} |_{2D} \xrightarrow{\cdot\sqrt{r} \quad ???} r^0 |_{3D} \quad (71)$$

or:

$$\frac{1}{r^2} |_{1D} \xrightarrow{\cdot r\sqrt{r}} \frac{1}{\sqrt{r}} |_{2D} \xrightarrow{\cdot r\sqrt{r} \quad ???} r |_{3D} \quad (72)$$

Integrating our hypothetical interdependences:

$$V_{3D} \sim r \quad (73)$$

or:

$$V_{3D} \sim r^2 \quad (74)$$

Once more putting the received and expected interdependences together:

$E(r)$	expected	received
1D	$\sim r$	$\sim r^0 \vee \sim \frac{1}{r}$
2D	$\sim \ln r$	$\sim \sqrt{r}$
3D	$\sim \frac{1}{r}$	$\sim r \vee \sim r^2$

## 5.2 Summary

Some model has been built, on which we imposed the suspicion that is appropriate for describing the Coulomb field. The results we received on this stage show a discord with what we had expected, but it is not absolutely sure that it is inappropriate. The final test would be the study of a 3D arrangement with the following assumptions:

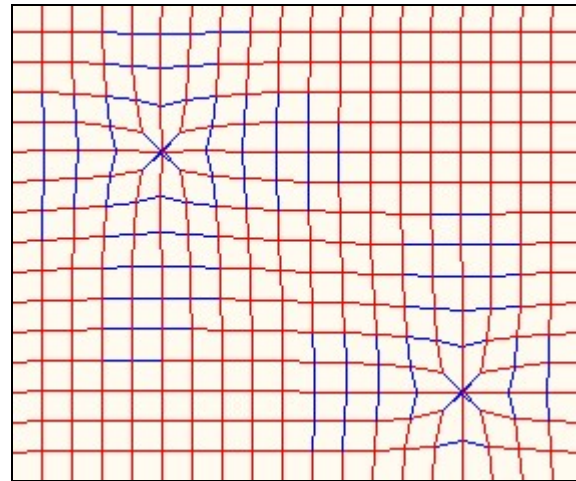
- a) the boundary conditions are ignored
- b) no directions are preferred

One of the most interesting conclusions of the simulation is a surprising similarity between the network arrangement and the visualization of field force lines for the Coulomb interactions (pic. 1). It is one of the reasons to continue the work on the model.

Finally we have to indicate the direction of future works on the subject:

1. a new, efficient algorithm
2. a new method of creating charges (adding and subtracting masses in the network)
3. more accurate definition of the charge
4. addition of springs connecting the masses with the background
5. placing additional springs at least with the closest neighbouring masses
6. gaining freedom from edges (a change of independent length to a non-zero value and giving the network an ability of a free movement)

Treating the electrostatic interaction in this quite simple (as we assume) classically-mechanical way seems interesting and still very promising mostly because of its untypical assumptions and its attempt to use one branch of science to describe another. A question whether it is possible to model an electrostatic interaction by an arrangement of springs, remains unanswered.



**Pic 1.** Model of electrostatic field

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The problem shown in this paper is a subject of study of this year Polish Children's Fund Comitee on the Warsaw University Z.U.Głazek Group. The group, to which apart from the author of the paper belong also: Juliusz Stasiewicz and Lech Stawikowski is working under the guidance of professor Stanisław Głazek. Latest information about the works and some of the programs are available on the site [www.fuw.edu.pl/~zuglazek/](http://www.fuw.edu.pl/~zuglazek/). The autor of the work would like to thank Karolina Sołtys, Magdalena Gończyńska-Jussis and Kamil Kamiński for both merithotical and linguistical cooperation.

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